MODULE-2

* Stream function:

Consider a 2-dimensional steady flow equation in X-Y plane.

$$u\,dy - v\,dx = 0$$

$$\frac{dy}{dx} = \frac{v}{u} \quad \text{--- (1)}$$

Integrate eq (10) 1, we will get:

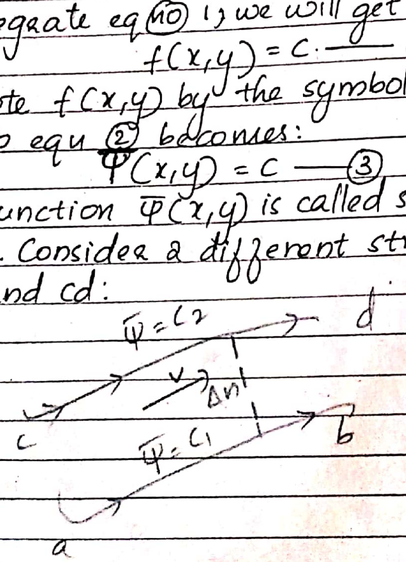
$$f(x,y) = C \quad \text{--- (2)}$$

Denote $f(x,y)$ by the symbol $\overline{\psi}$.

So equ (2) becomes:

$$\overline{\psi}(x,y) = C \quad \text{--- (3)}$$

The function $\overline{\psi}(x,y)$ is called stream function. Consider a different streamlines ab and cd:



Let Δn be the normal distance b/w ab and cd. The difference b/w $\overline{\psi}$ is the mass flow b/w 2 streamlines is defined per unit depth Ler to the page. Unit of $\overline{\psi}$ is $\frac{kg}{ms}$

$$\Delta\overline{\psi} = \rho A V$$

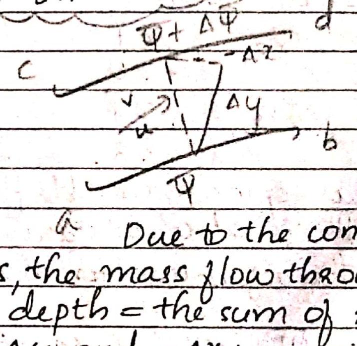
where, Area, $A = \Delta n \times 1$.

$$\boxed{\Delta\overline{\psi} = \rho V \Delta n} \quad \text{--- (4)}$$

$$\frac{\Delta\overline{\psi}}{\Delta n} = \rho V \quad \text{--- (5)}$$

$$\lim_{\Delta n \to 0} \frac{\Delta\overline{\psi}}{\Delta n} = \rho V.$$

$$\frac{\partial\overline{\psi}}{\partial n} = \rho V. \quad \text{--- (5)}$$



Due to the conservation of mass, the mass flow through Δn per unit depth = the sum of mass flow through Δy and $-\Delta x$ per unit depth.

So, mass flow $= \Delta\overline{\psi} = \rho V \Delta n$.

$$= \rho V (\Delta y - \Delta x)$$

$$\Delta\overline{\psi} = \rho u \Delta y - \rho v \Delta x.$$

Now we are assuming that Δn is very small, so cd approaches ab.

$$\therefore d\bar{\Psi} = \rho u dy - \rho v dx \quad (7)$$

$$d\bar{\Psi}(x,y) = \frac{\partial \bar{\Psi}}{\partial x} dx + \frac{\partial \bar{\Psi}}{\partial y} dy \quad (8)$$

Comparing (7) and (8); we get :

$$\rho u = \frac{\partial \bar{\Psi}}{\partial y}$$

$$\rho v = -\frac{\partial \bar{\Psi}}{\partial x}$$

///ly

In polar coordinates

$$\rho v_r = \frac{1}{r}\frac{\partial \bar{\Psi}}{\partial \theta}$$

$$\rho v_\theta = -\frac{\partial \bar{\Psi}}{\partial r}$$

In the case of incompressible flow 'ρ' is constant; $\bar{\Psi}$ is also constant and define a new stream function for incompressible flow 'Ψ'.

$$\therefore \frac{\bar{\Psi}}{\rho} = \Psi \cdot \left(\frac{kg\, m^3}{ms\, kg} = \frac{m^2}{s} \right)$$

Unit is m^2/s.

$$u = \frac{\partial \Psi}{\partial y}$$

$$v = -\frac{\partial \Psi}{\partial x}$$

In the case of Incompressible flow for polar coordinates.

$$V_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta}$$

$$V_\theta = -\frac{\partial \Psi}{\partial r}$$

* Advantages of stream function

(i) $\bar{\Psi}$ = constant or Ψ = constant gives the equation of streamline.

(ii) The flow velocities 'u' and 'v' can be obtained.

* Velocity potential function:

For an irrotational flow the vorticity is zero.

$$\xi = \nabla \times V = 0 \quad (1)$$

If 'ϕ' is a scalar function then the curl of the gradient of a scalar function is zero.

$$\nabla \times (\nabla . \phi) = 0 \quad (2)$$

Compare eq no (1) & (2):

$$V = \nabla . \phi \quad (3)$$

For irrotational flow their exist a scalar function ϕ such that the velocity is given by the gradient of ϕ. ϕ is denoted as velocity potential.

$$\phi = \phi(x,y,z) = \phi(r,\theta,z) = \phi(r,\theta,\phi)$$

Consider cartesian coordinates in eq no (3)

$$V = \nabla . \phi$$

$$ui + vj + wk = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

The coefficient of like unit vectors must be the same on both sides of equation, thus in cartesian coordinates:

$$u = \frac{\partial \phi}{\partial x}$$

$$v = \frac{\partial \phi}{\partial y} \text{ and}$$

$$w = \frac{\partial \phi}{\partial z}.$$

///rly in cylindrical coordinates:

$$V_r = \frac{\partial \phi}{\partial r}, \quad V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \quad V_z = \frac{\partial \phi}{\partial z}.$$

$$V_r = \frac{\partial \phi}{\partial r}, \quad V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \quad V_\phi = \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \phi}$$

Irrotational flows can be described by the velocity potential ϕ. Such flows are called potential flow.

* Difference b/w Ψ and ϕ.

Stream function	Velocity Potential
(i) Stream function is used for both rotational & irrotional.	(i) velocity function is only applicable for irrotional flow.
(ii) Stream functions defined for 2d flows only.	(ii) velocity potential function applicable for 3-D flows.
(iii) Flow field velocities are obtained by differentiating Ψ in the same direction.	(iii) Flow field velocities are obtained by differentiating ϕ normal to the velocity direction.

Irrotational flow:

A given field of flow may be entirely free of vortices or vorticity i.e., the vorticity may be zero everywhere in the field. Such a flow is said to be irrotational. If the vorticity is zero the angular velocity is also zero.

ie, $\omega_x = \omega_y = \omega_z = 0$

So $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$, $\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$

In the case of irrotational flow, velocity potential function (ϕ) will be constant.

$V = \nabla\phi$

$\phi = c$

$d\phi = 0$

$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$.

$u\,dx + v\,dy = 0$.

$\left(\frac{dy}{dx}\right)_{\phi = con} = -\frac{u}{v}$ —— ①

Stream function, $\psi = c$.

$\Rightarrow d\psi = 0$

$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$.

$-v\,dx + u\,dy = 0$

$\left(\frac{dy}{dx}\right)_{\psi = con} = \frac{v}{u}$ —— ②

Multiplying ① & ②, we get:

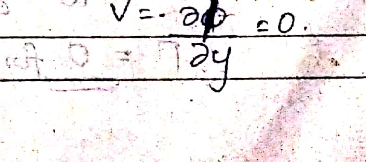
$\left(\frac{dy}{dx}\right)_{\phi = con} \times \left(\frac{dy}{dx}\right)_{\psi = con} \Rightarrow -\frac{u}{v} \times \frac{v}{u}$

$= -1$.

$\therefore$ Stream function and velocity potential function will be perpendicular to each other.

* Uniform flow:

Consider an uniform flow with velocity V_∞ oriented in the +ve x direction.



Velocity potential for this uniform flow is defined as: $u = \frac{\partial \phi}{\partial x} = V_\infty$ —— ①

$v = \frac{\partial \phi}{\partial y} = 0$.

from ① $\Rightarrow \partial\phi = V_\infty \partial x$

By Integrating: $\boxed{\phi = V_\infty x}$

In ~~cylindrical~~ polar coordinates $\phi = V_\infty r\cos\theta$

From the definition of stream function

$$u = \frac{\partial\psi}{\partial y} = V_\infty \quad \text{—②}$$

$$v = -\frac{\partial\psi}{\partial x} = 0$$

from ② $\Rightarrow \partial\psi = V_\infty dy$

By integrating : $\boxed{\psi = V_\infty y}$

In polar coordinates $\psi = V_\infty r\sin\theta$

Circulation, $\Gamma = -\oint_c v\,ds$

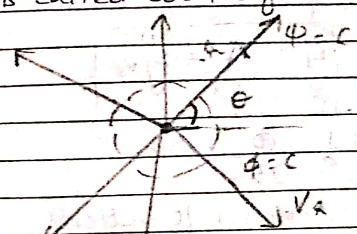
$$= -\oint_c V_\infty ds$$

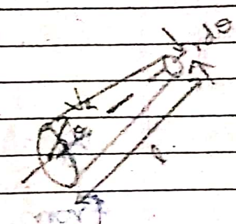
$$= V_\infty - \oint_c ds.$$

$\therefore \Gamma = 0$ for uniform flow.

* Source flow:

Consider a 2 dimensional incompressible flow where all streamlines are straight from a central point 'O'. Such a flow is called source flow.





Here the velocities are V_r and V_θ ie, V_r = radial velocity and V_θ = tangential velocity.

mass flowrate, $\dot{m} = \rho A V = \rho V_r r l d\theta$

$$= \int_0^{2\pi} \rho V_r r l d\theta$$

$$= 2\pi V_r r l \rho.$$

Volume flow rate, $\dot{V} = \frac{\dot{m}}{\rho} = 2\pi V_r r l$

Volume flow rate per unit length, $\dot{V}/l = \Lambda = 2\pi V_r r.$

$\Lambda = 2\pi V_r r \Rightarrow V_r = \frac{\Lambda}{2\pi r}$ and $V_\theta = 0$

$V_R = \frac{\Lambda}{2\pi R} = \frac{\partial\phi}{\partial R}$ (from velocity potential function)

$\partial\phi = \frac{\Lambda}{2\pi R}\,\partial R$

By Integrating:

$$\boxed{\phi = \frac{\Lambda}{2\pi} \ln R}$$

$V_\theta = \frac{1}{R}\frac{\partial\phi}{\partial\theta} = 0$

From stream function:

$V_R = \frac{1}{R}\frac{\partial\psi}{\partial\theta}$

$\partial\psi = V_R \times R \times \partial\theta$

$\int \partial\psi = \psi = \int \frac{\Lambda}{2\pi R} \times R \times \partial\theta$

$$\boxed{\psi = \frac{\Lambda}{2\pi}\theta}$$

$\Gamma = -\oint_C V_\infty \cdot ds$

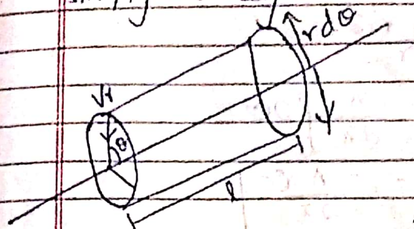
For evaluate the circulation for source flow, recall the $\nabla \times V = 0$ everywhere.

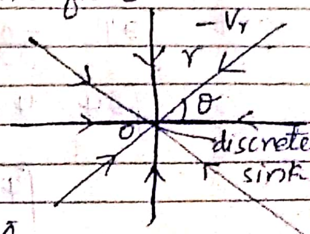
In turn,

$\Gamma = -\iint_S (\nabla \times V) \cdot ds = 0$

* Sink flow:

In a sink flow, the streamlines are directed towards the origin. For sink flow, the streamlines are still radial lines from a common origin, along which the flow velocity varies inversely with distance from point 'O'. Indeed, a sink flow is simply a negative source flow.





Mass flow rate, $\dot{m} = \int_0^{2\pi} -\rho V_r (r\,d\theta) l$

$\dot{m} = -\rho r l V_r \int_0^{2\pi} d\theta$

$\dot{m} = -2\pi r l \rho V_r$

$\dot{V} = \frac{\dot{m}}{\rho} = -2\pi r l V_R$

$\Lambda = \frac{\dot{V}}{l} = -2\pi r V_R$

$V_r = \frac{-\Lambda}{2\pi r}$

$V_r = \frac{\partial\phi}{\partial r} = \frac{-\Lambda}{2\pi r}$

$$\partial\phi = -\frac{\Lambda}{2\pi r}\partial r$$

$$\int \partial\phi = \phi = -\frac{\Lambda}{2\pi}\int \frac{1}{r}\,dr$$

$$\boxed{\phi = -\frac{\Lambda}{2\pi}\ln r}$$

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{-\Lambda}{2\pi r}$$

$$\partial\psi = \frac{-\Lambda}{2\pi \cancel{r}}\cancel{r}\,\partial\theta$$

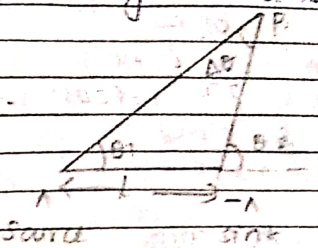
$$\int\partial\psi = \psi = \frac{-\Lambda}{2\pi}\int\partial\theta$$

$$\boxed{\psi = -\frac{\Lambda\theta}{2\pi}}$$

To evaluate the circulation for ~~source~~ sink flow recall the $\nabla\times V = 0$ everywhere. In turn, $\boxed{\Gamma = -\iint_S (\nabla\times V)\cdot ds = 0}$

* Doublet:

The degenerative case of a source sink pair that leads to a singularity is called a doublet. Consider a source of strength Λ and a sink strength $-\Lambda$ separated by a distance 'l'.



At any point P in the flow the stream function is:

$$\psi = \psi_{source} + \psi_{sink}$$

$$\psi = \frac{\Lambda\theta_1}{2\pi} - \frac{\Lambda\theta_2}{2\pi} \qquad [\Delta\theta = \theta_2 - \theta_1]$$

$$= \frac{-\Lambda\,\Delta\theta}{2\pi}$$

Let 'l' approaches 0 while the absolute magnitudes of the strength of the source and sink increases in such a fashion that the product Λl remains constant. In the process we obtain a doublet.

$$\Lambda l = K$$

$$\psi = -\frac{\Lambda\, d\theta}{2\pi}$$

$$\sin d\theta = \frac{a}{b}$$

$$d\theta = \frac{l\sin\theta}{r - l\cos\theta} \quad [\because \sin d\theta \simeq d\theta]$$

$$= \lim_{l\to 0} -\frac{\Lambda}{2\pi}\,\frac{l\sin\theta}{r - l\cos\theta}$$

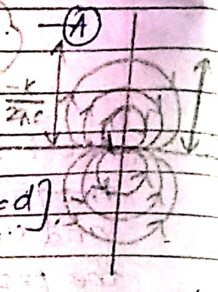
$$\psi = -\frac{K}{2\pi}\cdot\frac{\sin\theta}{r}$$

|||ly $\phi = \frac{K}{2\pi}\,\frac{\cos\theta}{r}$

ψ is a constant.

$$\therefore -\frac{K}{2\pi}\,\frac{\sin\theta}{r} = C.$$

$$r = \frac{-K\sin\theta}{2\pi C} \quad \text{—(A)}$$



$$r = y\sin\theta.$$

$$r = d\sin\theta \quad [\because y = d] \quad \text{—(B)}$$

From (A) and (B)

$$\frac{-K}{2\pi C}\sin\theta = d\sin\theta$$

$$\therefore d = \frac{-K}{2\pi C}$$ [−ve sign indicates that ψ flows]

* Vortex flow:

It is a flow where all the streamlines are concentric circles about a given point. Velocity will be constant in a streamline but it vary from one streamline to another inversely with distance from the common centre.

$$V_r = 0.$$

$$V_\theta = \frac{C}{r}. \qquad 2\pi r$$

$$\Gamma = -\oint v\,ds = -V_\theta \oint ds.$$

$$\Gamma = -V_\theta 2\pi r$$

$$= -2\pi r V_\theta.$$

$$V_\theta = \frac{-\Gamma}{2\pi r}$$

In polar coordinates, velocity potential,

$$V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}.$$

$$\frac{-\Gamma}{2\pi r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

By integrating, $\phi = \frac{-\Gamma\theta}{2\pi}$.

Stream function:

$$V_\theta = -\frac{\partial \psi}{\partial r}$$

$$\therefore -\frac{\partial \psi}{\partial r} = \frac{-\Gamma}{2\pi r}$$

By integrating: $\psi = \frac{\Gamma}{2\pi} \log r$

Circulation: It is a flow in which $\nabla \times V = 0$ at every point except origin.

$$V_\theta = \frac{C}{r} \quad \text{—(A)}$$

$$V_\theta = \frac{-\Gamma}{2\pi r} \quad \text{—(B)}$$

(A) = (B)

$$\frac{C}{r} = \frac{-\Gamma}{2\pi r} \Rightarrow \boxed{\Gamma = -2\pi C} \quad \text{—(C)}$$

$$\boxed{\Gamma = -\iint_S (\nabla \times V)\, ds} \quad \text{—(D)}$$

$$\text{(C)} = \text{(D)} \Rightarrow -\iint_S (\nabla \times V)\, ds = -2\pi C$$

$\because$ at origin $r = 0$, & $ds = 0$; $|\nabla \times V|\, ds = 2\pi C$

$$|\nabla \times V| = \frac{2\pi C}{ds}$$

At origin $r = 0$ and $ds = 0$

$$\therefore \boxed{|\nabla \times V| = \infty}$$

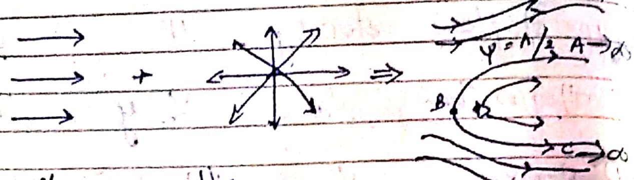
Vortex flow is irrotational everywhere except at the point $r = 0$ where the vorticity is infinite.

* Type of flow	velocity	ψ	ϕ
(i) Uniform flow	V_∞	$V_\infty y$	$V_\infty x$
(ii) Source	$V_r, V_\theta = 0$	$\psi = \frac{\Lambda}{2\pi} \theta$	$\phi = \frac{\Lambda}{2\pi} \log r$
(iii) Doublet	(V_r, V_θ)	$-\frac{K}{2\pi} \frac{\sin\theta}{r}$	$\frac{K}{2\pi} \frac{\cos\theta}{r}$
(iv) Vortex flow	$(V_r = 0, V_\theta)$	$\frac{\Gamma}{2\pi} \ln r$	$-\frac{\Gamma}{2\pi} \theta$
(v) Sink flow	$V_r, V_\theta = 0$	$-\frac{\Lambda}{2\pi} \theta$	$-\frac{\Lambda}{2\pi} \ln r$

	Circulation
uniform flow	0
source flow	0
Sink flow	0
Vortex flow	$-2\pi C$

* Combination of Uniform flow with Source and sink:

Consider a polar co-ordinate system with source strength "Λ" located at the origin. Superimpose on this flow a uniform stream with velocity V_∞ moving from left to right.



uniform flow | source flow | [~~Flow over a~~ semi-infinite body]

ψ = the stream function

$$\psi = V_\infty r\sin\theta + \frac{\Lambda}{2\pi}\theta = \text{constant} \quad \text{—(1)}$$

$\psi_U = V_\infty r\sin\theta$

$\psi_s = \frac{\Lambda}{2\pi}\theta$

source is located @ O.

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad V_\theta = -\frac{\partial\psi}{\partial r}$$

$$V_r = \frac{1}{r}\left[V_\infty r\cos\theta + \frac{\Lambda}{2\pi}\right] = \boxed{V_\infty\cos\theta + \frac{\Lambda}{2\pi r}} \quad \text{—(2)}$$

$$V_\theta = -V_\infty\sin\theta \quad \text{—(3)}$$

$\frac{\Lambda}{2\pi r}$ is the radial velocity from the source.

$V_\infty\cos\theta$ is the component of free stream velocity in radial direction.

$V_\theta = -V_\infty\sin\theta$ is the tangential velocity at stagnation point B, $V_r = 0$ & $V_\theta = 0$ & $\theta = \pi$

$$V_\theta = 0$$

$$-V_\infty\cancel{\sin\theta}^{0} = 0$$

$$V_r = 0$$

$$V_\infty\cos\theta + \frac{\Lambda}{2\pi r} = 0$$

$$V_\infty\cos\pi + \frac{\Lambda}{2\pi r} = 0$$

$$-V_\infty + \frac{\Lambda}{2\pi r} = 0$$

$$r = \frac{\Lambda}{2\pi V_\infty} \qquad \boxed{r = \frac{\Lambda}{2\pi V_\infty}} \quad \text{—(4)}$$

Stagnation point located at $(r,\theta) = \left(\frac{\Lambda}{2\pi V_\infty}, \pi\right)$

Substituting the value of r & θ in (1)

$$\psi = V_\infty \times \frac{\Lambda}{2\pi V_\infty} \sin\pi + \frac{\Lambda}{2\pi}\pi = \text{constant}$$

$$\psi = \frac{\Lambda}{2} = \text{constant}$$

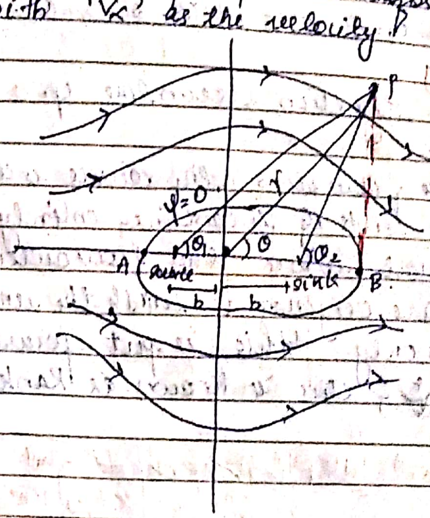
Stagnation point is a distance $\frac{\Lambda}{2\pi V_\infty}$ directly upstream of the source.

Value of r decrease if V_∞ increases.

Hence the streamline that goes through the stagnation point is described as $\psi = \frac{\Lambda}{2}$. This streamline is shown as curve ABC.

If we take a sink of equal strength as the source then the resulting body shape will be closed. consider a polar co-ordinate system with a source & sink placed at a distance b to the left & right of the origin respectively.

The strength of the source & sink are Λ & −Λ. In addition superimpose a uniform stream with 'V_∞' as the velocity.



$\theta_1 = \tan^{-1}\frac{y}{x+b}$

$\theta_2 = \tan^{-1}\frac{y}{x-b}$

The stream function for the combined flow at any point P with coordinates (r,θ)

$$\psi = \psi_{uni} + \psi_{sou} + \psi_{sink}$$

$$= V_\infty r\sin\theta + \frac{\Lambda}{2\pi}\theta_1 - \frac{\Lambda}{2\pi}\theta_2$$

$$\psi = V_\infty r\sin\theta + \frac{\Lambda}{2\pi}(\theta_1 - \theta_2)$$

$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$ & $V_\theta = -\frac{\partial\psi}{\partial r} \Rightarrow V_r = V_\infty\cos\theta + \frac{\Lambda}{2\pi r}(\theta)$

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A & B are stagnation points at

$A = \theta = \theta_1 = \theta_2 = \pi$. at $B = \theta = \theta_1 = \theta_2 = 0$.

for stagnation stream line $\psi = 0$

all the flow from the source is consumed by the sink & is contained entirely inside the oval where the flow outside the oval has originated with the uniform stream only. This is put forward by Rankine & this is known as Rankine oval